Math 2010 Week 12

Lagrange Multipliers

Q When does f have global extrema subject to constraint $g=c$?

A sufficient condition:

- The level set $S=\{g=c\}$ is closed and bounded
- f is continuous on S

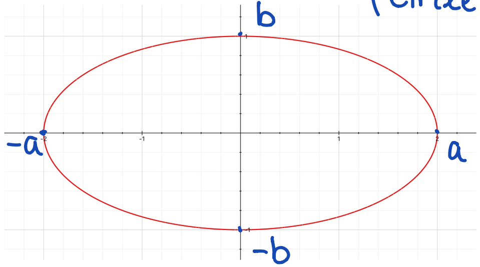
By EVT, f has global extrema on S.

Quadratic Constraint for 2-variable (Conic Section)

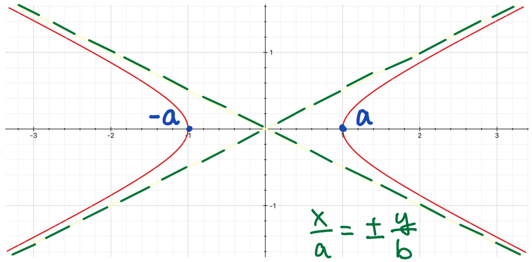
$$g(x,y)=Ax^2+2Bxy+Cy^2+Dx+Ey+F$$

Some typical examples of $g=c$:

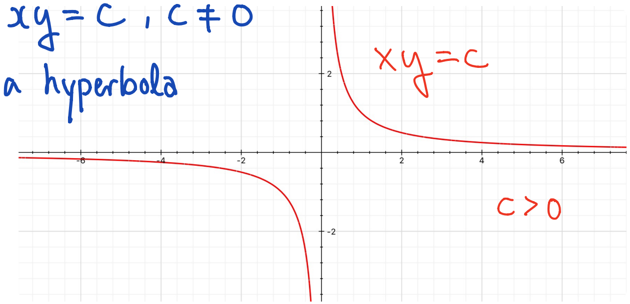
(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a,b>0$ (Ellipse, Circle if $a=b$)



(ii) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ a,b>0$ (Hyperbola)

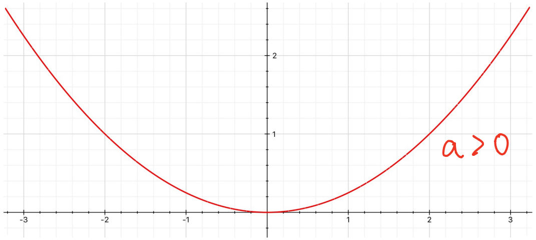


Rmk $xy=c,\ c\neq 0$ also a hyperbola



(iii) $y = ax^2$, $a \neq 0$ (Parabola)

(only 1 quadratic term)



(iv) Degenerate Cases

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0 \rightsquigarrow$ a point $(0,0)$
- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1 \rightsquigarrow$ empty set
- $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0 \rightsquigarrow \dfrac{x}{a} = \pm \dfrac{y}{b}$

  a pair of intersecting lines

- $x^2 = c \rightsquigarrow x = \pm\sqrt{c}$

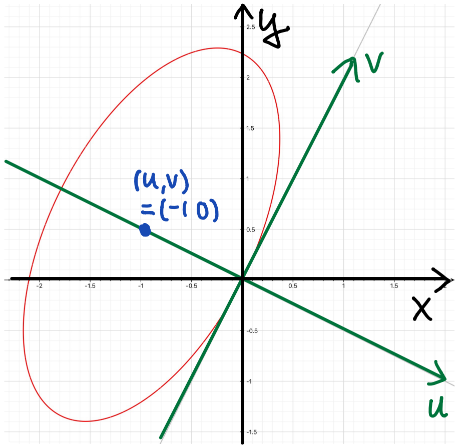
  a pair of parallel lines (double line if $c=0$)

Fact By a change of coordinates, any quadratic constraint $g(x,y) = c$ can be transformed to one of the forms above:

$\Rightarrow$ Ellipse, Hyperbola, Parabola, Degenerate

eg $17x^2 - 12xy + 8y^2 + 16\sqrt{5}x - 8\sqrt{5}y = 0$

$$\Leftrightarrow \frac{(u+1)^2}{1^2} + \frac{v^2}{2^2} = 1, \text{ where } u = \frac{2x-y}{\sqrt{5}} \quad v = \frac{x+2y}{\sqrt{5}}$$



Rmk In the last example, u and v are chosen so that the u-axis $\perp$ v-axis.

Such u and v can be found using theory of symmetric matrices in linear algebra

Among the non-degenerate quadratic constraints above, only ellipse is closed and bounded.

Any continuous $f(x,y)$ restricted to an ellipse has both global max/min

It is not true for hyperbola and parabola :

A continuous $f(x,y)$ restricted to a hyperbola or parabola may not have global max/min.

Quadratic Constraint for 3-variable

$$g(x,y,z) = Ax^2 + By^2 + Cz^2 + 2Pxy + 2Qyz + 2Rzx + Dx + Ey + Fz + G$$

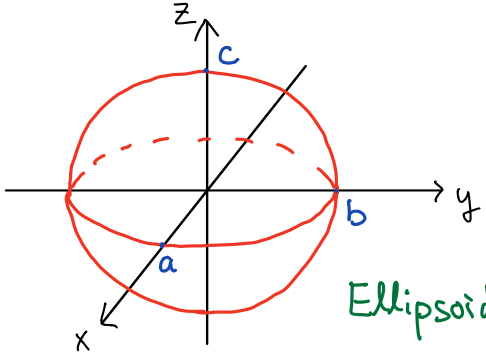
Some typical examples of $g = c$

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, $a,b,c>0$

How to graph it ?

$x^2 + y^2 + z^2 = 1$ is a unit sphere

Rescale in each coordinate direction



Ellipsoid

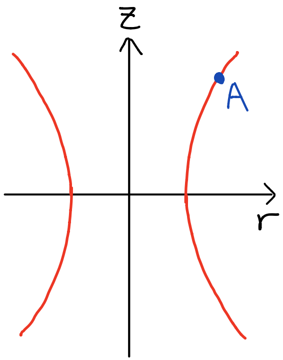
- Graph $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Up to rescaling, can assume $a=b=c=1$

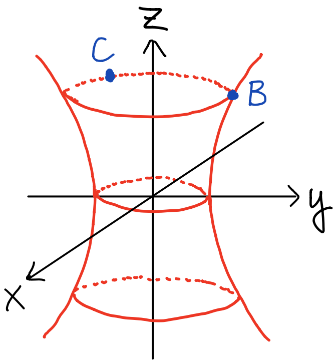
$$\rightsquigarrow \quad x^2 + y^2 - z^2 = 1$$

Let $r = \sqrt{x^2+y^2}$

$\quad = $ distance from (x,y,z) to z-axis



$\rightsquigarrow$ Revolve around z-axis



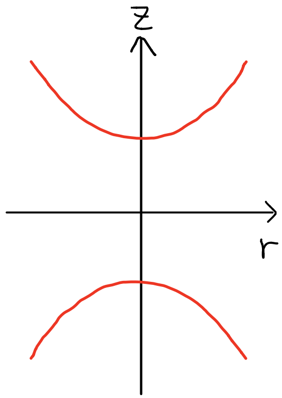
$r^2 - z^2 = 1$ (✱) — Hyperbola

$x^2+y^2-z^2 = 1$ (✱✱) — Hyperboloid of 1 sheet

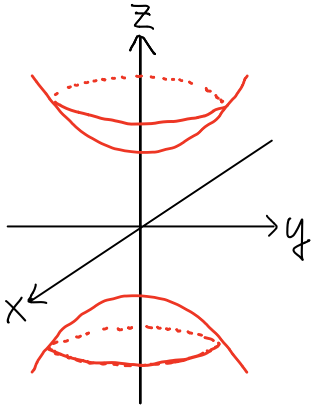
<u>Rmk</u> $A = (r,z) = (5, \sqrt{24})$ on (✱)

$\Rightarrow$ $B = (0, 5, \sqrt{24})$, $C = (-3, -4, \sqrt{24})$ on (✱✱)

- $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$



$\rightsquigarrow$ Revolve around z-axis



$r^2 - z^2 = -1$ — Hyperbola

$x^2+y^2-z^2 = -1$ — Hyperboloid of 2 sheets

<u>Ex</u> Graph

- $x^2+y^2-z^2 = 0$ (Elliptical cone)
- $z = x^2+y^2$ (Elliptical Paraboloid)
- $z = x^2 - y^2$ (Hyperbolic Paraboloid)

Graphs of standard quadratic surfaces

Source : philschatz.com

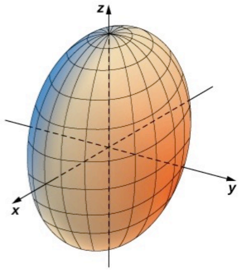
Characteristics of Common Quadric Surfaces

Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Traces
In plane $z=p$: an ellipse
In plane $y=q$: an ellipse
In plane $x=r$: an ellipse

If $a=b=c$, then this surface is a sphere.

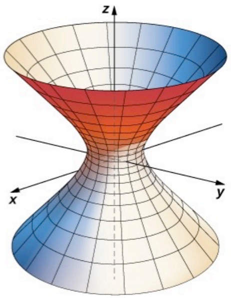


Hyperboloid of One Sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Traces
In plane $z=p$: an ellipse
In plane $y=q$: a hyperbola
In plane $x=r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.

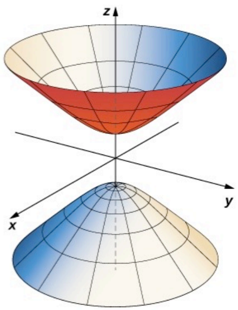


Hyperboloid of Two Sheets

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Traces
In plane $z=p$: an ellipse or the empty set (no trace)
In plane $y=q$: a hyperbola
In plane $x=r$: a hyperbola

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.



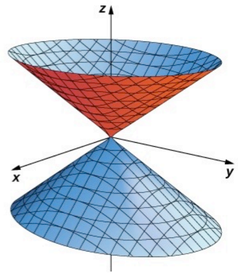
Characteristics of Common Quadric Surfaces

Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Traces
In plane $z=p$: an ellipse
In plane $y=q$: a hyperbola
In plane $x=r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin

The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.

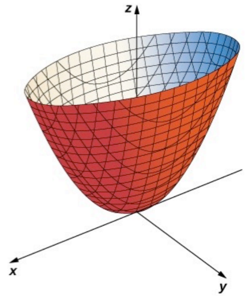


Elliptic Paraboloid

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Traces
In plane $z=p$: an ellipse
In plane $y=q$: a parabola
In plane $x=r$: a parabola

The axis of the surface corresponds to the linear variable.

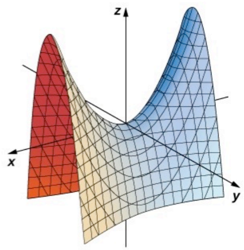


Hyperbolic Paraboloid

$$z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

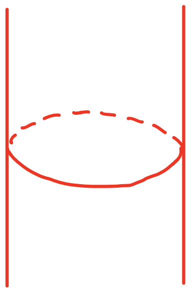
Traces
In plane $z=p$: a hyperbola
In plane $y=q$: a parabola
In plane $x=r$: a parabola

The axis of the surface corresponds to the linear variable.



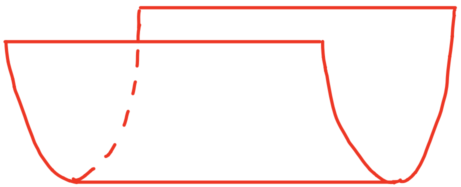
Besides the standard non-degenerate cases above, there are also degenerate cases, including cylinders of conic sections

eg



$x^2+y^2=1$

Cylinder of Ellipse



$z=x^2$

Cylinder of parabola

Similar to the case of 2-variable:

Any quadratic constraint $g(x,y,z)=c$ can be transformed to one of the standard forms by a change of coordinates.

Among the cases above, only ellipsoid is closed and bounded.

Any continuous $f(x,y,z)$ restricted to an ellipsoid has both global max/min

It is not the case for other quadratic surfaces.

Back to finding max / min under constraint.

eg Find the point on the ellipse

$$x^2 + xy + y^2 = 9$$

(Ex: Why?)

with maximum x-coordinate

Sol Let $f(x,y) = x$

$$g(x,y) = x^2 + xy + y^2$$

Maximize f under constraint $g = 9$

The ellipse $g = 9$ is closed and bounded.

f is continuous. By EVT, max. exists.

$$\nabla f = [1 \quad 0]$$

$$\nabla g = [2x+y \quad x+2y]$$

Note $\nabla g = [0 \quad 0] \iff (x,y) = (0,0)$

$(0,0)$ is not on the ellipse.

Use Lagrange Multipliers,

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 9 \end{cases} \Rightarrow \begin{cases} 1 = \lambda(2x+y) \cdots \text{(i)} \\ 0 = \lambda(x+2y) \cdots \text{(ii)} \\ x^2 + xy + y^2 = 9 \cdots \text{(iii)} \end{cases}$$

$$\text{(i)} \Rightarrow \lambda \neq 0$$

$$\therefore \text{(ii)} \Rightarrow x + 2y = 0 \Rightarrow x = -2y \cdots \text{(iv)}$$

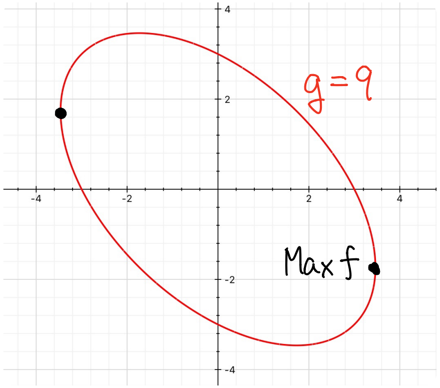
Put (iv) into (iii),

$$(-2y)^2 + (-2y)y + y^2 = 9 \Rightarrow 3y^2 = 9, \; y = \pm\sqrt{3}$$

By (iv), $(x,y) = (-2\sqrt{3}, \sqrt{3})$ or $(2\sqrt{3}, -\sqrt{3})$

Comparing x-coordinates,

Answer is $(2\sqrt{3}, -\sqrt{3})$



eg Find the point(s) on the hyperboloid $xy - yz - zx = 3$ closest to the origin.

Rmk Standard argument can be used to show closest point(s) exist.

However, the hyperboloid is unbounded $\Rightarrow$ farthest point does not exist.

Sol

Let $f(x,y,z) = \|(x,y,z) - (0,0,0)\|^2$

$$= x^2 + y^2 + z^2$$

Minimize f under constraint

$$g(x,y,z) = xy - yz - zx = 3$$

$\nabla f = [2x \quad 2y \quad 2z]$ $\quad \nabla g = [y-z \quad x-z \quad -x-y]$

Note $\nabla g \neq [0,0,0]$ on $g = 3$

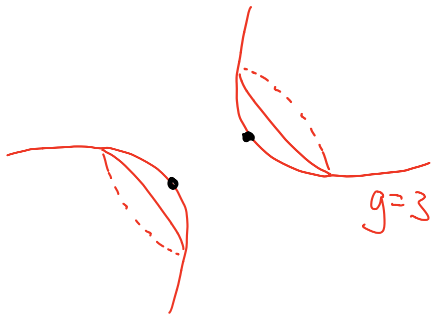
Use Lagrange Multipliers,

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 3 \end{cases} \underset{(Ex)}{\Longleftrightarrow} \begin{array}{c} (x,y,z) = \pm(1,1,-1) \\ \lambda = 1 \end{array}$$

Note $f(1,1,-1) = f(-1,-1,1) = 3$

$\therefore$ Closest points are $\pm(1,1,-1)$

Corresponding distance $= \sqrt{3}$



Lagrange Multipliers with multiple constraints

Let $f, g_1, g_2, \cdots, g_k$ be C^1 functions on $\Omega \subseteq \mathbb{R}^n$

$$S = \{ x \in \Omega : g_i(x) = c_i ,\ i = 1, \cdots, k \}$$

Suppose

1. a is a local extremum of f on S
2. $\nabla g_1(a), \cdots, \nabla g_k(a)$ are linearly independent

Then

$$\begin{cases} \nabla f(a) = \sum_{i=1}^{k} \lambda_i \nabla g_i(a) \text{ for some } \lambda_1, \cdots, \lambda_k \in \mathbb{R} \\ g_i(a) = c_i \text{ for } i = 1, \cdots, k \end{cases}$$

eg Maximize $f(x,y,z) = x^2 + 2y - z^2$

on the line L $\begin{cases} 2x - y = 0 \\ y + z = 0 \end{cases}$ in $\mathbb{R}^3$

It is given that f has maximum on L

Sol Let $g_1(x,y,z) = 2x - y$

$g_2(x,y,z) = y + z$

$\nabla f = [2x \quad 2 \quad -2z]$

$\nabla g_1 = [2 \quad -1 \quad 0]$ linearly

$\nabla g_2 = [0 \quad 1 \quad 1]$ independent

Use Lagrange Multipliers

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$$

Hence
$$\begin{cases} 2x = 2\lambda_1 + 0\lambda_2 & \cdots (1) \\ 2 = -\lambda_1 + \lambda_2 & \cdots (2) \\ -2z = 0\lambda_1 + \lambda_2 & \cdots (3) \\ 2x - y = 0 & \cdots (4) \\ y + z = 0 & \cdots (5) \end{cases}$$

(4), (5) $\Rightarrow$ $2x = y = -z$

(1), (3) $\Rightarrow$ $\lambda_1 = x \quad \lambda_2 = -2z$

(2) $\Rightarrow$ $-x - 2z = 2$

$\Rightarrow -x + 4x = 2 \Rightarrow x = \frac{2}{3}$

$\Rightarrow y = \frac{4}{3}, \ z = -\frac{4}{3}$

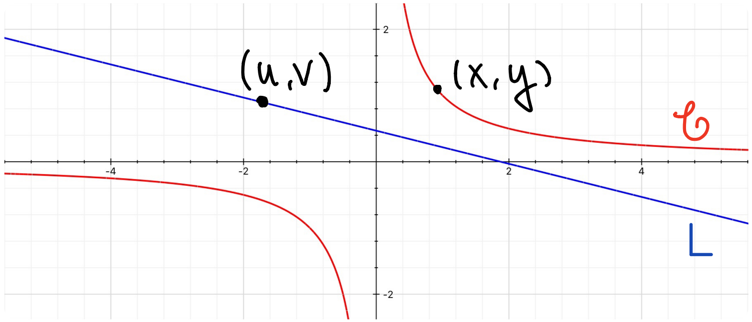
Since solution is unique and max. exists,

it must occurs at $(x, y, z) = \left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$

with max. value $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \frac{4}{3}$

eg Find the distance between

$\mathscr{C}: xy = 1$ and $L: x + 4y = \frac{15}{8}$



Sol Let $f(x, y, u, v) = \|(x,y) - (u,v)\|^2$

$= (x-u)^2 + (y-v)^2$

To find distance:

Minimize $f(x,y,u,v)$ under constraints

$g_1(x,y,u,v) = xy = 1$

$g_2(x,y,u,v) = u + 4v = \frac{15}{8}$

$$\nabla f = [\,2(x-u) \quad 2(y-v) \quad -2(x-u) \quad -2(y-v)\,]$$

$$\nabla g_1 = [\,y \quad x \quad 0 \quad 0\,]$$

$$\nabla g_2 = [\,0 \quad 0 \quad 1 \quad 4\,]$$

$\nabla g_1, \nabla g_2$ are lin. indept $\Leftrightarrow (x,y) \neq (0,0)$

But $xy=1 \Rightarrow \nabla g_1, \nabla g_2$ are lin. indept on $g_1 = 1$ and $g_2 = \frac{15}{8}$

Use Lagrange Multipliers

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 1 \\ g_2 = \frac{15}{8} \end{cases} \Rightarrow \begin{cases} 2(x-u) = \lambda_1 y \cdots (1) \\ 2(y-v) = \lambda_1 x \cdots (2) \\ -2(x-u) = \lambda_2 \cdots (3) \\ -2(y-v) = 4\lambda_2 \cdots (4) \\ xy = 1 \cdots (5) \\ u + 4v = \frac{15}{8} \cdots (6) \end{cases}$$

<u>Case 1</u> If $\lambda_1 = 0$ or $\lambda_2 = 0$, then

$x = u$, $y = v$

$(6) \Rightarrow x = \frac{15}{8} - 4y$

$(5) \Rightarrow \left(\frac{15}{8} - 4y\right) y = 1$

$$-4y^2 + \frac{15}{8} y - 1 = 0$$

No real solution

<u>Case 2</u> If $\lambda_1, \lambda_2 \neq 0$, then

$$\frac{1}{4} = \frac{x-u}{y-v} = \frac{y}{x} \Rightarrow x = 4y$$

$(5) \Rightarrow 4y^2 = 1 \Rightarrow y = \pm \frac{1}{2}$

$\therefore (x,y) = (2, \frac{1}{2})$ or $(-2, -\frac{1}{2})$

If $(x,y)=\left(2,\frac{1}{2}\right)$

$$\frac{2-u}{\frac{1}{2}-v}=\frac{1}{4} \Rightarrow 8-4u=\frac{1}{2}-v$$

$$\Rightarrow \begin{cases} -4u+v=-\frac{15}{2} \\ u+4v=\frac{15}{8} \end{cases}$$

$$\Rightarrow (u,v)=\left(\frac{15}{8},0\right)$$

If $(x,y)=\left(-2,-\frac{1}{2}\right)$

Similar calculation

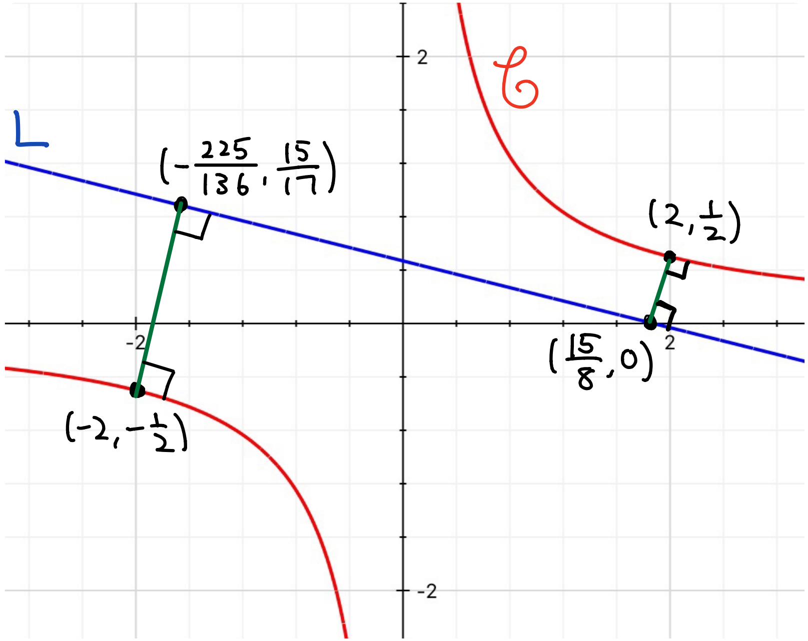
$$\Rightarrow (u,v)=\left(-\frac{225}{136},\frac{15}{17}\right)$$

Comparing the two solutions

f attains minimum at $(x,y,u,v)=\left(2,\frac{1}{2},\frac{15}{8},0\right)$

Distance between $\mathcal{C}$ and $L=\sqrt{f\left(2,\frac{1}{2},\frac{15}{8},0\right)}$

$$=\frac{\sqrt{17}}{8}$$

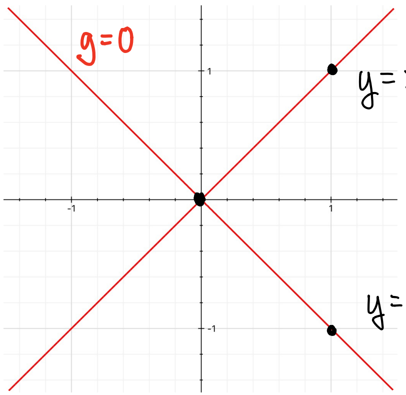


Implicit Function Theorem

Q When can we "solve" a constraint?

For example, if $g(x,y)=c$, can we find $y=h(x)$ such that $g(x,h(x))=c$?

eg1 Consider level set $g(x,y)=x^2-y^2=0$



$y=x$ near $(1,1)$ (i.e. $h(x)=x$)

$y=-x$ near $(1,-1)$ (i.e. $h(x)=-x$)

Near $(0,0)$, $y=x$? $y=-x$? or $\pm|x|$?

y is not uniquely determined by x

eg2 $S: x^2+y^2+z^2=2$ in $\mathbb{R}^3$

Q 3 variables, 1 equation $\Rightarrow$ S is 2-dim surface?

Solve for $z=h(x,y)$? $x=k(y,z)$?

We focus locally near $(0,1,1)$

If we can solve for z as a differentiable function $z=z(x,y)$ near $(0,1,1)$,

by implicit differentiation on $x^2+y^2+z^2=2$

$$\frac{\partial}{\partial x}: \; 2x+2z\frac{\partial z}{\partial x}=0$$

$$\frac{\partial}{\partial y}: \; 2y+2z\frac{\partial z}{\partial y}=0$$

$$\text{At } (x,y,z)=(0,1,1) \Rightarrow \begin{cases} 0+2\dfrac{\partial z}{\partial x}=0 \\ 2+2\dfrac{\partial z}{\partial y}=0 \end{cases}$$

$$\Rightarrow \left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right]=[0,-1] \text{ at } (0,1,1)$$

How about x as a differentiable function $x = x(y,z)$ near $(0,1,1)$?

If so, by implicit differentiation

$$\frac{\partial}{\partial y}: \quad 2x\frac{\partial x}{\partial y} + 2y = 0$$

$$\frac{\partial}{\partial z}: \quad 2x\frac{\partial x}{\partial z} + 2z = 0$$

coefficient of $\frac{\partial x}{\partial y}$ is $\frac{\partial g}{\partial x} = 0$

Put $(x,y,z) = (0,1,1) \Rightarrow \begin{cases} 0+2=0 \\ 0+2=0 \end{cases}$

Contradiction!

$\therefore$ x is not a differentiable function of y,z near $(0,1,1)$

Reason For $x^2+y^2+z^2=2$

If $y,z>1$ a little bit, no solution for x

If $y,z<1$ a little bit, 2 solutions for x

Let $g(x,y,z) = x^2+y^2+z^2$

Difference in the two cases:

At $(0,1,1)$ $\quad \frac{\partial g}{\partial z} = 2z \neq 0$

$$\frac{\partial g}{\partial x} = 2x = 0$$

In general, given constraint $F(x,y,z) = c$

If $z = z(x,y)$, then by implicit differentiation,

$$\left.\begin{aligned} \frac{\partial}{\partial x}: &\quad \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \\ \frac{\partial}{\partial y}: &\quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \end{aligned}\right\} (*)$$

If $F(\vec{a}) = c$, $\frac{\partial F}{\partial z}(\vec{a}) \neq 0$, then $(*)$ has solution

(No contradiction)

$\therefore$ $z = z(x,y)$ may exist and

$$\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = -\frac{1}{\frac{\partial F}{\partial z}(\vec{a})} \begin{bmatrix} \frac{\partial F}{\partial x}(\vec{a}) & \frac{\partial F}{\partial y}(\vec{a}) \end{bmatrix}$$

eg 3 (Multiple Constraints)

$$\mathcal{C}\ \begin{cases} x^2+y^2+z^2=2 & \text{3 variables} \\ x \quad + \quad z = 1 & \text{2 equations} \end{cases}$$

Q $\mathcal{C}$ is 1-dim curve? $y=y(x)$? $z=z(x)$?

If we can solve for y,z as differentiable functions $y(x)$, $z(x)$

Implicit Differentiation $\Rightarrow \begin{cases} 2x+2y\dfrac{dy}{dx}+2z\dfrac{dz}{dx}=0 \\ 1+\dfrac{dz}{dx}=0 \end{cases}$

$$\begin{bmatrix} 2y & 2z \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} -2x \\ -1 \end{bmatrix}$$

If this linear system has a solution, then $y=y(x)$, $z=z(x)$ may exist.

For instance, if $(x,y,z)=(0,1,1)$

$$\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

In general, given $F_1(x,y,z)=c_1$
$F_2(x,y,z)=c_2$

Suppose $F_i(a,b,c)=c_i$, $i=1,2$.

Do there exist differentiable functions $y=y(x)$, $z=z(x)$ near (a,b,c) such that

$$\begin{cases} F_1(x,y(x),z(x))=c_1 \\ F_2(x,y(x),z(x))=c_2 \end{cases} ?$$

If so, by implicit differentiation

$$\begin{cases} \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_1}{\partial y}\dfrac{dy}{dx} + \dfrac{\partial F_1}{\partial z}\dfrac{dz}{dx} = 0 \\[2ex] \dfrac{\partial F_2}{\partial x} + \dfrac{\partial F_2}{\partial y}\dfrac{dy}{dx} + \dfrac{\partial F_2}{\partial z}\dfrac{dz}{dx} = 0 \end{cases}$$

$$\begin{bmatrix} \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\[2ex] \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \end{bmatrix} \begin{bmatrix} \dfrac{dy}{dx} \\[2ex] \dfrac{dz}{dx} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial F_2}{\partial x} \\[2ex] -\dfrac{\partial F_2}{\partial x} \end{bmatrix}$$

If $\begin{bmatrix} \dfrac{\partial F_1}{\partial y}(\vec{a}) & \dfrac{\partial F_1}{\partial z}(\vec{a}) \\[2ex] \dfrac{\partial F_2}{\partial y}(\vec{a}) & \dfrac{\partial F_2}{\partial z}(\vec{a}) \end{bmatrix}^{-1}$ exists at $\vec{a} = (a,b,c)$

then $\begin{bmatrix} \dfrac{dy}{dx}(a) \\[2ex] \dfrac{dz}{dx}(a) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial F_1}{\partial y}(\vec{a}) & \dfrac{\partial F_1}{\partial z}(\vec{a}) \\[2ex] \dfrac{\partial F_2}{\partial y}(\vec{a}) & \dfrac{\partial F_2}{\partial z}(\vec{a}) \end{bmatrix}^{-1} \begin{bmatrix} -\dfrac{\partial F_2}{\partial x}(\vec{a}) \\[2ex] -\dfrac{\partial F_2}{\partial x}(\vec{a}) \end{bmatrix}$

Generally,

given $n+k$ variables

k equations

$$\begin{cases} F_1(x_1, \cdots, x_n, y_1, \cdots, y_k) = C_1 \\ \vdots \\ F_k(x_1, \cdots, x_n, y_1, \cdots, y_k) = C_k \end{cases}$$

When can $y_1, \cdots, y_k$ be expressed as functions of $x_1, \cdots, x_n$ locally?

Implicit Function Theorem